1 Abstract

This paper will detail my attempt to prove an unproved conjecture known as the Lonely Runner Conjecture. The original problem is thought to involve Diophantine approximation but this paper provides a different way but hardly efficient manner of viewing the problem. Several variations are placed on the problem and include but are not limited to restricting the speeds to be integers, rationals and finally irrationals or reals in general.

2 The Problem

Prove or disprove the following:

Conjecture: Suppose $K$ runners having distinct constant speeds start at a common point and run laps on a circular track with circumference $C$. Then for any given runner, there is a time at which that runner is distance at least $\frac{C}{K}$ (along the track) away from every other runner.

3 The Problem: Special Cases

The following theorems and corresponding proofs will attempt to show that there exists a time when all the runners are equally distributed around the circular path. Clearly, this
arrangement satisfies the conjecture but note it is not the only one.

**Theorem:** Given an integer circumference \( C \) and \( n \) runners with distinct constant integer speeds, then there exists an integer time at which all the runners are equally distanced along the track if and only if \( n \) divides \( C \) and some permutation of the speeds in modulo \( C \) is in arithmetic progression.

**Proof:**

For the sake of simplicity, we can assume that all the runners are moving counterclockwise from the initial position. Without loss of generality, denote the counterclockwise direction as positive. Then a runner running counterclockwise will have speed \( V \), some positive integer. Someone running at the same magnitude clockwise (the opposite direction) will have speed \(-V\). However, note that \(-V\) is equivalent to running at \( C - V \) in the counterclockwise direction. Hence, we can reorient each runner so that all of them are running in the same direction.

**Definition:** Let \( P_n(t) \) denote the position on the circular path of \( R_n \) (the \( n \)th runner) at time \( t \). Then \( P_n(t) = (V_1)(t) - \left\lfloor \frac{(V_n)(t)}{C} \right\rfloor \).

Now, we claim that it is a necessary condition that \( n \) divides \( C \). Consider the position of any arbitrary runner, \( R_k \) after an integer time interval. Then \( P_k(t) \), his position, is an integer since \((V_k)(t)\) is an integer and the floor function always produces an integer value. The difference between two integers is an integer. Then the positions of this runner’s immediate neighbors are \( P_k(t) \pm \frac{C}{n} \). Note that if \( n \) does not divide \( C \), then the positions of these neighbors are not integers. However, we just showed before that \( P_n(t) \) yields an integer output for integer \( V \) and \( t \). Hence, \( n \) must divide \( C \).

Now assuming that \( n \) does indeed divide \( C \), we still need to show that the theorem holds true iff the speeds are in arithmetic progression in modulo \( C \). Note that the position function, \( P_n(t) \) is equivalent to considering \((V_n)(t) \mod C\) for integer values of \( V, t \) and \( C \). Then without loss of generality we can permute \( V_1, V_2, \ldots V_n \) and relabel the first element \( V_1 \),...
the second $V_2$ and so on, so that $P_1(t) < P_2(t) < P_3(t) \ldots < P_n(t)$. In order for the runners to be equally distanced on the circle the following must hold:

\[ P_2(t) - P_1(t) = P_3(t) - P_2(t) \]
\[ P_3(t) - P_2(t) = P_4(t) - P_3(t) \]
\[ P_4(t) - P_3(t) = P_5(t) - P_4(t) \]
\[ \vdots \]
\[ P_1(t) - P_n(t) = P_n(t) - P_{n+1}(t) \]

This implies that

\[ (V_2)(t) - (V_1)(t) = (V_3)(t) - (V_2)(t) \mod C \]
\[ (V_3)(t) - (V_2)(t) = (V_4)(t) - (V_3)(t) \mod C \]
\[ (V_4)(t) - (V_3)(t) = (V_5)(t) - (V_4)(t) \mod C \]
\[ \vdots \]
\[ (V_1)(t) - (V_n)(t) = (V_n)(t) - (V_{n-1})(t) \mod C \]

Moreover, this means $2(V_i)t = (V_{i-1})(t) + (V_{i+1})(t)$. Dividing through by $t$, we get

\[ V_i = \frac{V_{i-1} + V_{i+1}}{2} \]

In other words, the set of speeds must be an arithmetic sequence!

**Theorem:** Given a circular path of integer circumference $C$ and $n$ people with **rational** speeds, $V_1, V_2 \ldots V_n$, there exists an integer time $t$ at which every runner is equally distanced around the circle if multiplication of the speeds by an integer scalar yields a new set of speeds that is an arithmetic progression modulo $C$.

**Proof:** Let $V_i = \frac{p_i}{q_i}$ where $gcd(p_i, q_i) = 1$. Then lets scale all these speeds by $gcd(q_1, q_2, \ldots q_n)$. The set of $V_i$ are all integers now and hence, we can check if it possible by using our
previous result. That is, if this set of integer speeds is in arithmetic sequence reduced modulo $C$. If it is, then we simply scale the amount of time by the same number we scaled the velocities by, that is the $gcd(q_1, q_2, \ldots, q_n)$.

**Theorem:** Given a circular path of integer circumference $C$ and $n$ people with *irrational* velocities $V_1, V_2, V_3, \ldots V_n$, there does not exist an integer time $t$ at which all the runners are equally distanced on the circle.

**Proof:** I claim it is impossible to achieve such a state of equally distributed runners.

Suppose there existed an integer time where the runners are distributed evenly. Then from the previous theorem we proved, these positions are rational if not integer. However, not that it is impossible to reach a rational position when one has an irrational velocity unless the time interval is irrational which is clearly not the case. Hence, this situation ceases to exist.

## 4 Conclusion

This paper has done little to prove or disprove the Lonely Runner Conjecture. Rather, it has examined a few specific cases that satisfy the parameters of the conjecture. We have shown that all the runners are evenly spaced (meaning they are exactly $C/K$ away from one another) when their speeds form an arithmetic series. However, this clearly does not cover all the possibilities.